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Optimal design of elastic rods: extension of a minimum energy solution

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Abstract

An optimal design problem solved previously for an elastic rod, which is a mixture of two materials and which hangs under its own weight is solved here for a different parameter range on the product of elastic stiffness and density of the materials. The problem is also considered when the parameters are at the junction of the two ranges and the possibility of an infinite number of solutions arises, raising questions of well posedness of the problem there. The method used is the maximum principle of Pontryagin and Hestenes. © 2001 Elsevier Science Ltd. All rights reserved.

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1. Introduction

The problem of designing an elastic rod stressed by its own weight as it hangs vertically from a fixed support has been solved for different choices of design variable and by different methods by Fosdick and Royer-Carfagni (1996a) (Fosdick et al. 1996; Warner, 2000). Fosdick and Royer-Carfagni consider the rod to be a mixture of two constituents with the volumetric concentration of the stiffer material as the design variable. The cross-sectional area is constant. They use methods of functional analysis and variational calculus which involve the construction of the lower convex envelope to the strain energy function in their solution. I use cross-sectional area of the rod as design variable and take the material properties to be fixed. The maximum principle (Pontryagin et al., 1962; Hestenes, 1980) is the method used for constructing candidates for the optimizer. Both approaches seek the minimum of the potential energy of the rod not only with respect to admissible functions for the axial displacement and its associated strain distribution, as in the ordinary equilibrium problem, but also with respect to the design variable. The latter quantity is subject to upper and lower bound constraints and to an isoperimetric constraint.

The solution found by both methods has essentially the same mathematical form with differing physical interpretation of the variables involved. The design variable is a bang-bang control, in the parlance of the

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control theorist, taking on only its least and greatest values. Depending on the parameters and boundary conditions of the problem, the optimal design consists of two or three regions in which these values alternate, with two different two-region solutions. The optimizer is unique at each choice of end displacement.

In finding this result, a fourth candidate three-region optimizer was ruled out (Warner, 2000, Appendix B.4). Although not explicitly considered in their work, this possibility does not arise for the concentration density problem of Fosdick and Royer-Carfagni (1996a) because of an inequality that the products of the stiffness and body force coefficients of their two pure materials are assumed to satisfy. That assumption is automatically satisfied if area is taken as the design variable. It is the purpose of this work to show when that fourth type of solution can arise in the mixture problem for the rod.

2. Statement of the minimization problem

A slight generalization of the notation used in the works mentioned above leads to the following statement of the minimization problem. Minimize the potential energy functional $\Pi(U(X), \varepsilon(X), C(X))$ defined by

$$\Pi(U(X), \varepsilon(X), C(X)) = \int_0^L \left\{ \frac{1}{2} S(C)\varepsilon^2 - B(C)U \right\} \mathrm{d}X \tag{2.1}$$

with respect to the three functions axial displacement U, axial strain ε, and a design variable C subject to the following constraints and conditions:

$$\text{(a)}\ \frac{\mathrm{d}U}{\mathrm{d}X} = \varepsilon(X), \qquad U(0) = 0, \qquad U(L) = \Delta, \tag{2.2}$$

$$\text{(b)}\ \Phi_1 \equiv C_1 - C \leqslant 0, \qquad \Phi_2 \equiv C - C_2 \leqslant 0, \tag{2.3}$$

$$\text{(c)}\ \int_0^L C(X)\mathrm{d}X = LC_0, \quad C_1 < C_0 < C_2, \tag{2.4}$$

(d) B is a twice differentiable positive function of C on the interval (C_1, C_2) (and later B is taken to be linear in C);

(e) S is a twice differentiable positive monotone increasing function of C on the interval (C_1, C_2); moreover,

$$\frac{\mathrm{d}^2}{\mathrm{d}C^2}\left(\frac{1}{S(C)}\right) > 0, \tag{2.5}$$

so that the strain energy density $S(C)\varepsilon^2/2$ is nonconvex in C and ε. The admissible sets of functions $\varepsilon(X)$ and $C(X)$ are piecewise continuous on $(0, L)$ while the admissible set for $U(X)$ consists of continuous functions with piecewise continuous first derivatives. U must also satisfy the boundary conditions in Eq. (2.2). The differential equation constraint in Eq. (2.2) holds where $\varepsilon(X)$ is continuous.

Note that the integral ("isoperimetric") constraint (2.4) cannot be satisfied by $C(X)$ identically equal to either of its bounds. Note further that $C(X) = C_0$ is an admissible function and is the only constant value that satisfies the integral constraint. Also, for dimensional consistency in Π with the differential equation constraint of Eq. (2.2) taken into account,

$$\dim(S) \times \dim(U) = \dim(B) \times (\dim(X))^2.$$

For the rod, the stiffness function $S = EA$ and the body force function $B = \rho g A$. Warner (2000) fixes the material, i.e. takes Young's modulus E and the mass density ρ as constant, and takes A as the $C(X)$ of functional (2.1). Fosdick and Royer-Carfagni (1996a) fix the geometry, i.e., A constant and let the mixture material properties E and ρ depend on the concentration density c, $0 \leqslant c(x) \leqslant 1$, of the stiffer of two materials. The density is the linear convex combination of the densities of two pure materials:

$$\rho(c) = \rho_0(1-c) + \rho_1 c,$$

while $E(c)$ satisfies conditions like those placed on $S(C)$ above. The assumption is made that $E_0\rho_0 < E_1\rho_1$ in order to construct the solution. In the present notation, that becomes in terms of the products of the stiffness and body force coefficient values at the bounding values on C,

$$S_1B_1 < S_2B_2.$$

This is automatically satisfied by the choice of A as the design variable with the material properties ρ and E constant, since the product SB equals ρgEA^2. However, the inequality ordering either way can occur for mixtures or alloys of real materials. Without suggesting that one would actually make alloys of these elements, lead, silver, and titanium come in that order from least to greatest Young's modulus. The product of modulus and density for lead is less than that for silver but the product for lead is greater than that for titanium.

The approach to be used is essentially the same as that in Warner (2000), Section 3 and Appendix B, and so many details of the argument will be left out here. As there, the calculations are simplified by nondimensionalizing. Change from $\{X, C(X), S(C), B(C), U(X), \varepsilon(X), \Pi(U, \varepsilon, C)\}$ to $\{x, c(x), s(c), b(c), u(x), \eta(x), \pi(u, \eta, c)\}$ by the choices:

(1) $X = Lx$, so that $0 \leqslant x \leqslant 1$;
(2) $C(X) = (C_2 - C_1)c(x) + C_1$, so that $0 \leqslant c(x) \leqslant 1$ and

$$\int_0^1 c(x)\,\mathrm{d}x = c_0 = \frac{C_0 - C_1}{C_2 - C_1}, \quad 0 < c_0 < 1; \tag{2.6}$$

(3) $S(C) = \hat{S}s(c)$, where $\hat{S}$ is the weighted harmonic mean of the end values $S_1 \equiv S(C_1)$ and $S_2 \equiv S(C_2)$ of $S(C)$:

$$\frac{1}{\hat{S}} = \frac{1-c_0}{S_1} + \frac{c_0}{S_2}; \tag{2.7}$$

(4) $B(C) = \bar{B}b(c)$, where $\bar{B}$ is the weighted arithmetic mean of the end values $B_1 \equiv B(C_1)$ and $B_2 \equiv B(C_2)$ of $B(C)$ (and not necessarily $B(C_0)$, though it will be later when B is taken to be linear in C):

$$\overline{B} = (1-c_0)B_1 + c_0B_2; \tag{2.8}$$

(5) $U(X) = \overline{B}L^2/\hat{S}\,u(x)$ and $\Delta = \overline{B}L^2/\hat{S}\,\delta$;
(6) $\varepsilon(X) = \overline{B}L/\hat{S}\,\eta(x)$, so that $\mathrm{d}u/\mathrm{d}x = \eta$; and
(7) $\Pi(U, \varepsilon, C) = \overline{B}^2L^3/\hat{S}\pi(u, \eta, c)$, where

$$\pi(u(x), \eta(x), c(x)) = \int_0^1 \left\{\frac{1}{2}s(c)\eta^2 - b(c)u\right\}\mathrm{d}x. \tag{2.9}$$

The choices of $\hat{S}$ and $\overline{B}$ in Eqs. (2.7) and (2.8) are not arbitrary but are strongly suggested by the solution process. Indeed, with no nondimensionalization at all one would be led to a division by the harmonic mean combination on the right side of Eq. (2.7) in solving for the end forces in the two-interval candidate optimizers discussed later. This harmonic mean is also at the heart of the construction by Fosdick and Royer-Carfagni of the lower convex envelope of the strain energy density function. Similar weighted-mean constructions appear in bounding estimates for effective moduli of composites (Christensen, 1979, chapter IV). Homogenization theory for mixtures generally leads to such estimates.

The problem becomes the minimization of functional (2.9) with respect to $u(x)$, $\eta(x)$, and $c(x)$ subject to the following constraints:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \eta(x), \tag{2.10a}$$

$$u(0) = 0, \qquad u(1) = \delta, \tag{2.10b}$$

$$\phi_1 \equiv -c(x) \leqslant 0, \qquad \phi_2 \equiv c(x) - 1 \leqslant 0, \tag{2.10c}$$

$$\int_0^1 c(x)\,\mathrm{d}x = c_0, \quad 0 < c_0 < 1. \tag{2.10d}$$

The functions $s(c)$ and $b(c)$ are twice differentiable functions in [0, 1]. The function $s(c)$ is positive, monotone increasing, and satisfies the condition for nonconvexity of $s(c)\eta^2/2$:

$$s > 0, \qquad \frac{\mathrm{d}s}{\mathrm{d}c} > 0, \qquad \frac{\mathrm{d}^2}{\mathrm{d}c^2}\left(\frac{1}{S}\right) > 0. \tag{2.11}$$

Its end values $s_1 \equiv s(0)$ and $s_2 \equiv s(1)$ have weighted harmonic mean equal to 1:

$$\frac{1-c_0}{s_1} + \frac{c_0}{s_2} = 1. \tag{2.12a}$$

The function $b(c)$ is positive and the weighted arithmetic mean of the end values $b_1 \equiv b(0)$ and $b_2 \equiv b(1)$ is equal to 1:

$$(1-c_0)b_1 + c_0 b_2 = 1. \tag{2.12b}$$

For future use, note that by rewriting this last relation as

$$\frac{1-c_0}{s_1} b_1 s_1 + \frac{c_0}{s_2} b_2 s_2 = 1 \tag{2.12c}$$

and using Eq. (2.12a), one obtains the result that, if $b_1 s_1 = b_2 s_2$, then they must equal 1; if they are unequal, one must be greater than 1 and the other one between 0 and 1. It also must be true that $s_1 < 1 < s_2$, $\min(b_1, b_2) < 1 < \max(b_1, b_2)$.

3. Necessary conditions for an optimizer

Now apply the Pontryagin–Hestenes maximum principle to generate, first, the necessary conditions that an optimizer $\{u^*(x), \eta^*(x), c^*(x)\}$ must satisfy and, second, candidate optimizers satisfying those conditions. The principle as stated in Hestenes (1980, Chapter 6, Theorem 4.1), for "control problems of Lagrange with inequality constraints" requires the existence of multipliers

$$\lambda_0 \geqslant 0, \quad \lambda, \quad p(x), \quad \mu_1(x), \quad \mu_2(x), \tag{3.1a}$$

not vanishing simultaneously on the closed interval $0 \leqslant x \leqslant 1$ and a "pre-Hamiltonian" function $H(u, p, \eta, c, \mu_1, \mu_2)$ defined by

$$H = p\eta - \lambda_0\left(\frac{1}{2}s(x)\eta^2 - b(c)u\right) - \lambda c - \mu_1\phi_1 - \mu_2\phi_2, \tag{3.1b}$$

such that the following conditions hold:

(i) The multipliers $\mu_1(x)$, $\mu_2(x)$ are continuous on each interval of continuity of $\eta^*(x)$, $c^*(x)$. Moreover, they are nonnegative. Each is zero wherever the corresponding function ϕ_1 or ϕ_2 (Eq. (2.10c)) is strictly less than zero.

(ii) The multiplier $p(x)$ is continuous and satisfies, with $u^*(x)$, $\eta^*(x)$, $c^*(x)$ and the 2 μs, the given differential equation constraint (2.10a)

$$\frac{du}{dx} = \frac{\partial H}{\partial p} = \eta(x) \tag{3.2a}$$

and the additional equations

$$\frac{dp}{dx} = \frac{\partial H}{\partial u} = -\lambda_0 b(c), \tag{3.2b}$$

$$\frac{dH}{d\eta} = p - \lambda_0 s(c)\eta = 0, \tag{3.2c}$$

$$\frac{\partial H}{\partial c} = \mu_1 - \mu_2 - \lambda - \lambda_0\left(\frac{1}{2}s'(c)\eta^2 - b'(c)u\right) = 0 \tag{3.2d}$$

on each interval of continuity of $\eta^*(x)$ and $c^*(x)$.

(iii) The function H evaluated on a minimizer is continuous on the closed interval $0 \leqslant x \leqslant 1$ and satisfies the relation

$$\frac{dH}{dx} = \frac{\partial H}{\partial x} = 0 \tag{3.3}$$

on each interval of continuity of $\eta^*(x)$ and $c^*(x)$.

(iv) The inequality

$$H(u^*, p^*, \eta, c, 0, 0) = p^*\eta - \lambda_0\left(\frac{1}{2}s(c)\eta^2 - b(c)u^*\right) - \lambda c \leqslant H(u^*, p^*, \eta^*, c^*, 0, 0) = \sup\ H = \overline{H} \tag{3.4}$$

holds for all admissible controls $\eta(x)$ and $c(x)$ on the left side and for $u^*(x)$, $\eta^*(x)$, $c^*(x)$, together with the solution values of the multipliers $p^*(x)$, λ_0, λ elsewhere on both sides of Eq. (3.4). The constant $\overline{H}$ is the value of the supremum of H guaranteed by the result Eq. (3.3).

4. The program for using these conditions

The program to be followed from here consists of the following steps:

(1) Show that the choice $\lambda_0 = 0$ would lead to all multipliers vanishing, which is not allowed. Then λ_0 can be taken as 1 since it scales all other multipliers and H. This argument is exactly the same as that in Warner (2000, Appendix B.1), and will not be repeated here.

(2) Show the supremum of the function on the left of inequality (3.4), or rather of its suboptimal form with the * functions (unknown a priori) replaced by functions $u(x)$, $p(x)$ treated as known, leads to the choice of $c(x)$ as either 0 or 1 if b is linear in c. Satisfying the integral constraint, leads to the result that the total lengths (measures) $\xi_1$and ξ_2 at each of the bounds must satisfy

$$\xi_2 = 1 - \xi_1 = c_0.$$

Appendix A.1 gives the details that differ from Warner (2000, Appendix B.2).

(3) By the use of the solution forms for any segment with constant c value, show that any optimal sequence of three segments at alternating c values must have $p(x)$ and $\eta(x)$ zero at the middle of the central

segment (except for special parameter choices that give $b_1 s_1 = b_2 s_2 = 1$). Since $p(x)$ is a continuous monotone decreasing function, this suffices to show that an optimizer must have not more than three segments (Appendix A.2, again for parts of the argument differing from the previous paper).

(4) Construction of the four candidates for global optimizer and their associated $\overline{H}$, $\mu(x)$, and π quantities is carried out next. The two three-region candidates are shown to be mutually exclusive with one or the other not possible as an optimizer depending on the *sb*-product inequality (Appendix A.3). For each ordering, the proof of the previous paper goes through with final proof of global optimality based on comparison of the values of the functional π for the candidates (Appendix A.4).

(5) Study of the exceptional case $b_1 s_1 = b_2 s_2 = 1$, leads to an infinite multiplicity of optimizers, all having the same $\overline{H}$ and π values (Appendix A.5).

One should verify that the multiplier functions $\mu(x)$ that must be positive are indeed so, though the proof of optimality through the π values obviates the need for this. Again, the procedure to be followed is discussed in the previous paper.

A summary of the final results follows. It comes in three parts, corresponding to three different choices of the ratio $b_1 s_1 / b_2 s_2$ of the products of body force and stiffness coefficients for the pure materials 1 and 2. In the first two parts, the results are further determined by the choice of the prescribed end displacement δ.

(A) Suppose $b_1 s_1 / b_2 s_2 < 1$.
If $|\delta| < \delta_1 = \xi_1 / 2 s_1$, then the three-region design with $c(x) = (0, 1, 0)$ in intervals of lengths $\{\xi_1/2 + s_1\delta, \xi_2, \xi_1/2 - s_1\delta\}$ is the sole optimal design.
If $\delta \geqslant \delta_1$, then the two-region design $c(x) = \{0, 1\}$ in intervals of lengths $\{\xi_1, \xi_2\}$ is optimal.
If $\delta \leqslant -\delta_1$, then the two-region design $c(x) = \{1, 0\}$ in intervals of lengths $\{\xi_2, \xi_1\}$ is optimal.
(B) Suppose $b_1 s_1 / b_2 s_2 > 1$.
If $|\delta| < \delta_2 = \xi_2 / 2 s_2$, then the three-region design with $c(x) = (1, 0, 1)$ in intervals of lengths $\{\xi_2/2 + s_2\delta, \xi_1, \xi_2/2 - s_2\delta\}$ is the sole optimal design.
If $\delta \geqslant \delta_2$, then the two-region design $c(x) = 1, 0$ in intervals of lengths $\{\xi_2, \xi_1\}$ is optimal.
If $\delta \leqslant -\delta_2$, then the two-region design $c(x) = 0, 1$ in intervals of lengths $\{\xi_1, \xi_2\}$ is optimal.
(C) If $b_1 s_1 / b_2 s_2 = 1$, an infinite family of multiregion solutions are optimal for all δ, all having the same Hamiltonian and π values. Not only are both two-region solutions and limiting forms of the three-region solutions given above optimal, but also all multiregion solutions satisfying equilibrium and the measure conditions arising from the integral constraint.

Part A is simply a restatement in the present notation of the result found before in the papers of Fosdick and Royer-Carfagni (1996a) and Warner (2000); part B extends that result to the new parameter case; and part C raises questions about the well posedness of the optimization problem for those parameter values.

5. Conclusions and further investigations

The formal rod problem for a two-material mixture solved here is the simplest possible problem admitting of an analytical solution over the full range of material parameter values. The basic result is that minimum energy occurs for a design that alternates the pure materials in the precise way detailed in parts A and B of the proposition of the last section, with the exceptional case of part C not yet fully understood.

This is not the only mixture problem that has such a solution. Fosdick and Royer-Carfagni (1996b) have also shown by their function space methods that a sphere, solid or hollow, made of two isotropic elastic materials and given a uniform radial displacement at its outer surface has minimum strain energy when the two pure materials alternate in a precise way in two or three regions depending on the displacement prescribed. This suggests that one should be able to prove a general theorem for minimizing an energy-like functional for two-material elastic bodies possessing sufficient symmetry in geometry and loading (e.g., rods, beams, cylinders, and spheres) so that only one independent variable (an axial or radial coordinate for the bodies mentioned) remains and where the energy is convex in one control variable, a generalized strain, but nonconvex in another, a design or concentration density variable. I believe that I have been able to do so recently.

Construction of analytical solutions for problems involving bodies other than rod or sphere would be useful for insights they might give. Is the not-more-than-three region result a general one? Is there any difference between the results for beams for which the equilibrium problem is governed by fourth order differential equations and the others, where second order systems occur?

Extension to problems where nonlinear material behavior or nonlinear kinematics occurs would be interesting. There is some evidence that the result still is true for elastic-like bodies where the curvature-displacement relation is nonlinear.

How these minimum energy states might occur physically as perhaps the end result of some sort of diffusion process in the mixture would be an important scientific advance. I don't know how to set up such a model. In any case, introducing a second time-like independent variable would take the problem out of the class treated by the standard maximum principle.

Appendix A. The calculations for the program of Section 4

The differences between the calculations in Appendix B of Warner (2000) and those necessary here arise from the slightly more general use of the stiffness and body force functions $s(c)$ and $b(c)$ rather than the single area variable $c = a$.

A.1. The optimal $c(x)$ must be piecewise constant when $b(c)$ is linear, taking on only its bounding values 0 and 1

Maximize

$$\tilde{H}(\eta, c) = p(x)\eta - \left(\frac{1}{2}s(c)\eta^2 - b(c)u(x)\right) - \lambda c \tag{A.1}$$

with respect to η and c, treating $p(x)$ and $u(x)$ (and λ) as known. As in Appendix B.2 of the previous paper, we find

$$\eta = p/s(c) \tag{A.2}$$

(the elasticity relation Eq. (3.2c) again). Substituting this value of η back into (A.1), we obtain the intermediate function $H^*(c)$:

$$H^*(c) = \tilde{H}\left(\frac{p}{s}, c\right) = \frac{p^2}{2s(c)} + b(c)u - \lambda c. \tag{A.3}$$

We now seek the supremum of the latter, subject to the inequality constraints on c.

The second derivative of H^* with respect to c is

$$\frac{\partial^2 H^*}{\partial c^2} = \frac{p^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}c^2}\left(\frac{1}{s(c)}\right) + u\frac{\mathrm{d}^2 b(c)}{\mathrm{d}c^2}. \tag{A.4}$$

Take $b(c)$ to be linear in c so that its second derivative is zero:

$$b(c) = (1 - c)b_1 + cb_1,\ b(c_0) = 1. \tag{A.5}$$

Then the nonconvexity condition (2.10c) on $s(c)$ will ensure that the supremum of H^* does not occur where the first derivative is zero. Thus, there is an end point maximum and $c(x)$ can take on only its bounding values 0 or 1. Since neither can be extended over the whole length and still satisfy the integral constraint, there must be at least two segments to an optimizer. The value or values of x where switching from one bound to the other must be determined.

Satisfaction of the integral constraint shows that the total lengths (measures) ξ_1 and ξ_2 for which $\phi_1 = 0$ and $\phi_2 = 0$, respectively, satisfy

$$\xi_2 = 1 - \xi_1 = c_0. \tag{A.6}$$

From here on ξ_1 and ξ_2 will be used rather than c_0 to emphasize the total lengths at each of the c values in an optimizer. Relations (2.12a) and (2.12b) then become

$$\frac{\xi_1}{s_1} + \frac{\xi_2}{s_2} = 1, \qquad b_1\xi_1 + b_2\xi_2 = 1. \tag{A.7}$$

The construction of the two two-region candidate optimizers will be carried out later.

A.2. The solution forms for segments of constant section: proof that no design with four or more regions can be optimal

Follow the same process as that given in Warner (2000, Appendix B.3) for finding the solution for any segment of the rod in which $c(x)$ (and so s and b) have constant values. One obtains as before, that $p(x)$ is linear in x and $u(x)$ is quadratic. Continuity of p and u at the join between a $c = 0$ segment and a $c = 1$ segment leads to the constant values of the Hamiltonian and the multiplier λ as functions of $\tilde{u} = u(\tilde{x})$, $\tilde{p} = p(\tilde{x})$:

$$\overline{H} = E_1 = E_2 - \lambda, \qquad \lambda = E_2 - E_1, \tag{A.8}$$

where

$$E_\alpha = b_\alpha \tilde{u} + \frac{\tilde{p}^2}{2s_\varepsilon}, \qquad \alpha = 1, 2, \tag{A.9}$$

so that

$$\lambda = (b_2 - b_1)\tilde{u} + \left(\frac{1}{s_2} - \frac{1}{s_1}\right)\frac{\tilde{p}^2}{2}. \tag{A.10}$$

The forms of the multiplier functions $\mu_\alpha(x)$ which must be positive if a design is to be optimal are obtained from the equation $\partial H/\partial c = 0$ (Eq. (3.2d)). The contribution of each segment to the potential energy $\pi(u, \eta, c)$ can also be calculated.

To show that an optimal design may not have more than three segments, an argument similar to that after Eq. (B.8) in Warner (2000) is used, based on the constancy of $\overline{H}$ and the monotonicity of $p(x)$. There is, however, an extra step required before the conclusion can be drawn. Denote the values of (x, u, p) at two successive joins by $(\tilde{x}, \tilde{u}, \tilde{p})$ and $(\tilde{x}', \tilde{u}', \tilde{p}')$. Using the equality of the Es (Eq. (A.9)) for the two outer regions and the constancy of E in the middle region of any sequence of three with alternating c-values, one obtains

$$\frac{1}{2}\left(\frac{1}{b_2 s_2} - \frac{1}{b_1 s_1}\right)\left(\tilde{p}^2 - \tilde{p}'^2\right) = 0. \tag{A.11}$$

If $b_1 s_1 \neq b_2 s_2$ this requires that $\tilde{p}'^2 = \tilde{p}^2$. But $p(x)$ is a continuous monotone decreasing function and this relation can occur, and occur at most once, only if $\tilde{p}' = -\tilde{p}$. Therefore the optimizing design cannot consist of more than three segments. Moreover, if there are three segments, the length of the middle segment must be ξ_2 for the 0–1–0 design and ξ_1 for the 1–0–1 design with the value of $p(x)$ equal to zero at the midpoint of the segment.

A.3. The four candidate optimizers; mutual exclusivity of cases III and IV

Next, list the candidates following the procedures used in Appendix B.4 of the previous paper. The four different candidates will be distinguished where necessary by roman numerals I–IV, where I is the two-region candidate with $c(x) = (0, 1)$, II is the two-region candidate with $c(x) = (1, 0)$, III is the three-region candidate with $c(x) = (0, 1, 0)$, and IV is the three-region candidate with $c(x) = (1, 0, 1)$. All the solution forms will not be given here, but only the values of the four Hamiltonians. These are needed in later discussion and may serve here as check points for the reader wishing to reproduce the full result.

The solutions for cases I and II exist for all values of the end displacement δ. One finds the Hamiltonian for case I as a function of δ:

$$\overline{H}^{\mathrm{I}}(\delta) = E_1^{\mathrm{I}} = \frac{1}{2s_1}\left[\delta + \frac{1}{2}\left(\frac{\xi_2}{s_2} + b_1\xi_1\right)\right]^2. \tag{A.12}$$

The arithmetic mean identity for b and the harmonic mean identity for s (Eq. (A.7)) must be used.

It is easy to show that the results for case II may be computed from those for case I by making the substitution of $1 - x$ for x and $-\delta$ for δ. One finds

$$\overline{H}^{\mathrm{II}}(\delta) = b_1\delta + \overline{H}^{\mathrm{I}}(-\delta) = b_1\delta + \frac{1}{2s_1}\left[\frac{1}{2}\left(\frac{\xi_2}{s_2} + b_1\xi_1\right) - \delta\right]^2. \tag{A.13}$$

To construct the candidate extremals with three regions, again follow the techniques of the previous paper by first finding the lengths ζ^{i}, ζ^{iii} of the first and third segments. One obtains in case III for the lengths and end forces

$$\zeta^{\mathrm{i}} = \frac{\xi_1}{2} + s_1\delta, \qquad \zeta^{\mathrm{iii}} = \frac{\xi_1}{2} - s_1\delta,$$

$$p^* = b_1 s_1\delta + \frac{1}{2}, \qquad p^{**} = b_1 s_1\delta - \frac{1}{2},$$

with the results for IV obtained by interchanging subscripts 1 and 2. These solutions exist only for a finite range of δ:

$$|\delta| \leqslant \frac{\xi_1}{2s_1} \equiv \delta_1 \tag{A.14}$$

for existence of the case III optimal candidate and (through the subscript interchange property)

$$|\delta| \leqslant \frac{\xi_2}{2s_2} \equiv \delta_2 \tag{A.15}$$

for existence of the case IV optimal candidate.

For case III, the Hamiltonian is

$$\overline{H}^{\mathrm{III}}(\delta) = E_1^{\mathrm{III}} = \frac{1}{2s_1}\left[b_1 s_1\delta + \frac{1}{2}\right]^2. \tag{A.16}$$

For case IV,

$$\overline{H}^{\mathrm{IV}}(\delta) = E_1^{\mathrm{IV}} = \frac{b_1 b_2 s_2}{2}\delta^2 + \frac{b_1}{2}\delta + \frac{b_1}{8}\left(b_1\xi_1 + \frac{\xi_2}{s_2}\right). \tag{A.17}$$

The first result is that the three-region candidates III and IV are mutually exclusive; that is, one of the two may always be ruled out depending on the values of the parameters. Calculate the value of the $\mu(x)$ function for each at the midpoint of its central section. A little algebra and use of the identities (A.7) gives

$$\mu^{\mathrm{III}}(\hat{x}_{\mathrm{III}}) = \frac{b_2\xi_2}{8s_1}(1 - b_1 s_1), \tag{A.18}$$

$$\mu^{\mathrm{IV}}(\hat{x}_{\mathrm{IV}}) = \frac{b_1\xi_1}{8s_2}(1 - b_2 s_2). \tag{A.19}$$

Thus, if $b_1 s_1 < 1 < b_2 s_2$ then $\mu^{\mathrm{IV}}(\hat{x}_{\mathrm{IV}})$ is always less than zero, and IV cannot be a global optimizer; if $b_2 s_2 < 1 < b_1 s_1$ then the other three-region candidate III cannot be an optimizer: $\mu^{\mathrm{III}}(\hat{x}_{\mathrm{III}}) < 0$. Therefore, there are just three candidates to consider no matter which of these two cases of parameter ordering is involved.

Comparison of the $\overline{H}$ values for I and II also leads easily to a general result. From (A.12) and (A.13),

$$\overline{H}^{\mathrm{II}}(\delta) - \overline{H}^{\mathrm{I}}(\delta) = \left(\frac{\xi_2}{s_2}\right)\frac{(b_1 s_1 - 1)}{s_1}\delta. \tag{A.20}$$

Thus, when $b_1 s_1 < 1 < b_2 s_2$, I is to be preferred to II for $\delta > 0$ and II to I for $\delta < 0$, with the reverse preferences when $b_2 s_2 < 1 < b_1 s_1$.

The same difficulties mentioned in Warner (2000, Appendix B.4) arise here in attempting to complete the proof by arguing through the signs of the $\mu(x)$ functions. Rather it is best to make a direct comparison of the values of the energy functionals in each case.

A.4. Calculations of the four energy functionals π and proof of global optimality

Compute the values of the energy functionals π for the four cases. The four results are:

$$\pi^{\mathrm{I}}(\delta) = \frac{1}{2}\delta^2 - \frac{1}{2}\left(b_2\xi_2 + \frac{\xi_1}{s_1}\right)\delta - \frac{1}{24}\left\{4\frac{\xi_2}{s_2}\frac{\xi_1}{s_1} + \left(b_2\xi_2 - \frac{\xi_1}{s_1}\right)^2\right\}, \tag{A.21}$$

$$\pi^{\mathrm{II}}(\delta) = \pi^{\mathrm{I}}(-\delta) - (b_2\xi_2 + b_1\xi_1)\delta = \pi^{\mathrm{I}}(-\delta) - \delta, \tag{A.22}$$

$$\pi^{\mathrm{III}}(\delta) = \frac{b_1 s_1}{2}\delta^2 - \frac{1}{2}\delta - \frac{1}{24}\left[\frac{\xi_1}{s_1} + (b_2\xi_2)\left(\frac{\xi_1}{s_1}\right) + (b_2\xi_2)^2\right], \tag{A.23}$$

$$\pi^{\mathrm{IV}}(\delta) = \frac{b_2 s_2}{2}\delta^2 - \frac{1}{2}\delta - \frac{1}{24}\left[\frac{\xi_2}{s_2} + (b_1\xi_1)\left(\frac{\xi_2}{s_2}\right) + (b_1\xi_1)^2\right]. \tag{A.24}$$

Now, compute the energy differences between the three-region and the two-region candidates. For parameter choices $b_1 s_1 < 1 < b_2 s_2$ compare I and II with III. First, take the difference in the π values for I and III and rewrite it as a function of $\delta - \delta_1$ by adding and subtracting appropriate quantities and again using the arithmetic and harmonic mean identities (A.7). One finds

$$\pi^{\mathrm{I}}(\delta) - \pi^{\mathrm{III}}(\delta) = \frac{1 - b_1 s_1}{2}(\delta - \delta_1)^2. \tag{A.25}$$

Therefore, the energy for case I is always higher than that for case III for those values of δ where they both exist, and equals π^{III} only at the point where III fails to exist for positive δ and I becomes the optimizer.

For II and III, we would expect a similar result based on the other limiting point $-\delta_1$ and that is what we find:

$$\pi^{\mathrm{II}}(\delta) - \pi^{\mathrm{III}}(\delta) = \frac{1 - b_1 s_1}{2}(\delta + \delta_1)^2. \tag{A.26}$$

Thus, III has a lower energy than II and is to be selected where it exists. Finally, note that the difference between these results gives the difference between the energies for I and II:

$$\pi^{\mathrm{I}}(\delta) - \pi^{\mathrm{II}}(\delta) = \frac{1 - b_1 s_1}{2}(\delta - \delta_1)^2 - \frac{1 - b_1 s_1}{2}(\delta + \delta_1)^2 = -2(1 - b_1 s_1)\delta_1 \delta. \tag{A.27}$$

When $\delta > 0$, I has a lower energy than II and so is the global optimizer for $\delta \geqslant \delta_1$; II has lower energy than I and is confirmed as the optimizer for $\delta \leqslant -\delta_1$.

For the I–II–IV comparisons when $b_2 s_2 < 1 < b_1 s_1$, essentially the same computations will show that

$$\pi^{\mathrm{I}}(\delta) - \pi^{\mathrm{IV}}(\delta) = \frac{1 - b_2 s_2}{2}(\delta + \delta_2)^2, \tag{A.28}$$

$$\pi^{\mathrm{II}}(\delta) - \pi^{\mathrm{IV}}(\delta) = \frac{1 - b_2 s_2}{2}(\delta - \delta_2)^2, \tag{A.29}$$

reflecting the reversed roles of the I and II solutions in this parameter range. The I–II difference is now

$$\pi^{\mathrm{I}}(\delta) - \pi^{\mathrm{II}}(\delta) = \frac{1 - b_2 s_2}{2}(\delta + \delta_2)^2 - \frac{1 - b_2 s_2}{2}(\delta - \delta_2)^2 = 2(1 - b_2 s_2)\delta_2 \delta. \tag{A.30}$$

Therefore, I is the global optimizer when $\delta \leqslant -\delta_1 = -\xi_2/2s_2$, IV is the optimizer for $|\delta| < \delta_2$, and II is the optimizer for $\delta \geqslant \delta_1$. This completes the proof of parts A and B of the proposition at the end of Section 4.

A.5. The special parameter choice $b_1 s_1 = b_2 s_2 = 1$

If in the final results for the values of $\overline{H}$ and π for the four cases above the value 1 is substituted for $b_1 s_1 = b_2 s_2$, the result will be that all four $\overline{H}$'s have the same value and also all four πs:

$$\overline{H}(\delta) = \frac{1}{2s_1}\left[\delta + \frac{1}{2}\right]^2 = \frac{b_1}{2}\left[\delta + \frac{1}{2}\right]^2, \tag{A.31}$$

$$\pi(\delta) = \frac{1}{2}\delta^2 - \frac{1}{2}\delta - \frac{1}{24}. \tag{A.32}$$

This suggests that all four cases give optimizers for all δ values for which they exist. There is, however, more to be said.

The calculations for cases I and II under the additional condition $b_2 s_2 = b_1 s_1 = 1$ proceed exactly as the earlier construction of the solution functions. One finds that the end values of $p(x)$ must be the same for both:

$$p^* = \delta + \tfrac{1}{2}, \qquad p^{**} = \delta - \tfrac{1}{2}. \tag{A.33}$$

It follows that the $\overline{H}$ and π values are the ones given above for both cases at each δ.

The calculations for cases III and IV are complicated now by the fact that we can no longer prove that the value of $p(x)$ at the midpoint of the middle segment is zero, since the argument from Eq. (A.11) fails. Certainly there is a solution with $p = 0$ at that point and it is easy to verify that it too leads to a solution with the same end values of p^* and p^{**} (A.33) and the same $\overline{H}$ and π formulas (A.31) and (A.32). But it is just as easy to assume that the switching points are at $x = \zeta$ and $x = \zeta + \zeta_\alpha$, $\alpha = 2$ or 1, with ζ in the range $(0, \zeta_\beta)$, $\beta = 1$ or 2, respectively, and solve the corresponding generalized cases III and IV – and find that all give the same $\overline{H}$ and same π values independently of ζ. There is thus a whole family of candidates with three segments. Even this is not the whole story, however; the possibility of more than three regions must be considered, keeping the total measure condition in mind.

An alternative approach to the constant section solutions of Eqs. (A.8)–(A.10) and their contribution to the additive functional π gives insight into this parameter case. Consider an interval $x^- < x < x^+$ in which $c(x)$ is constant and take the reference point $\hat{x}$ for Eqs. (A.8)–(A.10) to be at the end x^-. The solution functions for the interval and the contribution to π can be written as before. Now, divide the interval [0, 1] into any number N of subintervals of lengths $\zeta^k, k = 1, 2, \ldots, N$, and assign the values $c = 0$ and $c = 1$ with their corresponding $s(c)$ and $b(c)$ to these intervals alternately, starting with either value. For definiteness, choose 0 as the first value; one may show that the same results hold for the choice of 1 first by interchanging subscripts 1 and 2 in what follows. In each interval, the solution functions and the contribution to π may be found by using the continuity of p and u at the joins so that the p^+, u^+ for one interval is the p^-, u^- for the next. Start at $x = 0$ using the boundary condition $u(0) = u_1^- = 0$, carrying along $p_1 = p(0)$ to be determined. At the other end, the boundary condition is $u(1) = u_N^+ = \delta$, with p_N^+ unknown. The sequence of continuity results for $p(x)$ is as follows:

$$p_1^+ = p_1^- - b_1\zeta^1 = p_2^-,$$

$$p_2^+ = p_2^- - b_2\zeta^2 = p_1^- - b_1\zeta^1 - b_2\zeta^2 = p_3^-,$$

$$p_3^+ = p_3^- - b_1\zeta^3 = p_1^- - b_1(\zeta^1 + \zeta^3) - b_2\zeta^2 = p_4^-,$$

$$\vdots$$

$$p_N^+ = p_N^- - b_{1(\text{or}2)}\zeta^N = p_1^- - b_1\sum_{k\,\text{odd}}\zeta^k - b_2\sum_{k\,\text{even}}\zeta^k.$$

The requirement that the integral constraint be met means that

$$\sum_{k\,\text{odd}}\zeta^k = \xi_1, \qquad \sum_{k\,\text{even}}\zeta^k = \xi_2,$$

and so the overall equilibrium condition

$$p_N^+ = p_1^- - 1 \tag{A.34}$$

is satisfied.

Now look at the conditions for continuity of $u(x)$. Here the special condition $b_2 s_2 = b_1 s_1 = 1$ requires that

$$u^+ = u^- + \frac{1}{2}\left[(p^-)^2 - (p^+)^2\right] \tag{A.35}$$

for each interval. Putting these special results together from one interval to the next, one may show that

$$u_N^+ = \delta = \frac{1}{2}\left[(p_1^-)^2 - (p_N^+)^2\right].$$

Together with Eq. (A.34) this determines the end values for any such design:

$$p_1^- = p(0) = \delta + \frac{1}{2}, \qquad p_N^+ = p(1) = \delta - \frac{1}{2}.$$

Calculating the energy functional with $bs = 1$ is simple, since all the p^3 terms at the interior nodes cancel from one interval to the next leaving the result

$$\pi = \frac{(p_1^-)^3 - (p_N^+)^3}{3} - E_1 \sum_{k\,\mathrm{odd}} \zeta^k - E_2 \sum_{k\,\mathrm{even}} \zeta^k.$$

But

$$E_2 = b_2\left(u_2^- + \frac{(p_2^-)^2}{2b_2s_2}\right) = b_2\left(u_1^+ + \frac{(p_1^+)^2}{2}\right) = b_2\left(u_1^- + \frac{(p_1^-)^2}{2}\right) = \frac{b_2}{2}\left(\delta + \frac{1}{2}\right)^2,$$

and

$$E_1 = b_1\left(u_1^- + \frac{(p_1^-)^2}{2}\right) = \frac{b_1}{2}\left(\delta + \frac{1}{2}\right)^2.$$

Therefore,

$$E_1 \sum_{k\,\mathrm{odd}} \zeta^k + E_2 \sum_{k\,\mathrm{even}} \zeta^k = \frac{b_1\xi_1 + b_2\xi_2}{2}\left(\delta + \frac{1}{2}\right)^2 = \frac{1}{2}\left(\delta + \frac{1}{2}\right)^2$$

and $\overline{H}$ and π are the same (Eqs. (A.31) and (A.32)) for all equilibrium designs satisfying the integral constraint. These results raise the question of well posedness of the design problem for such parameter values.

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